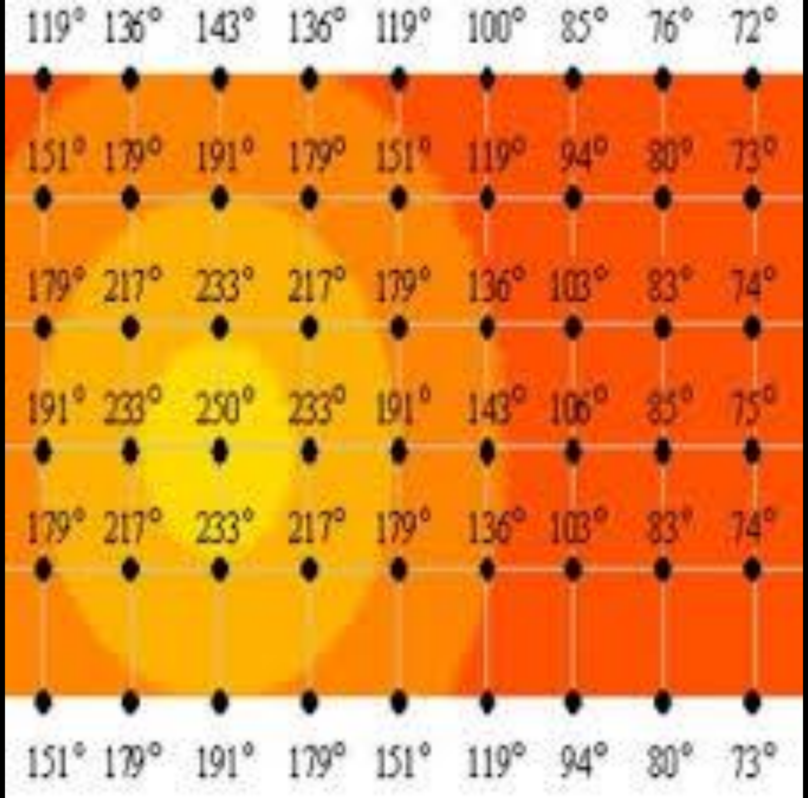
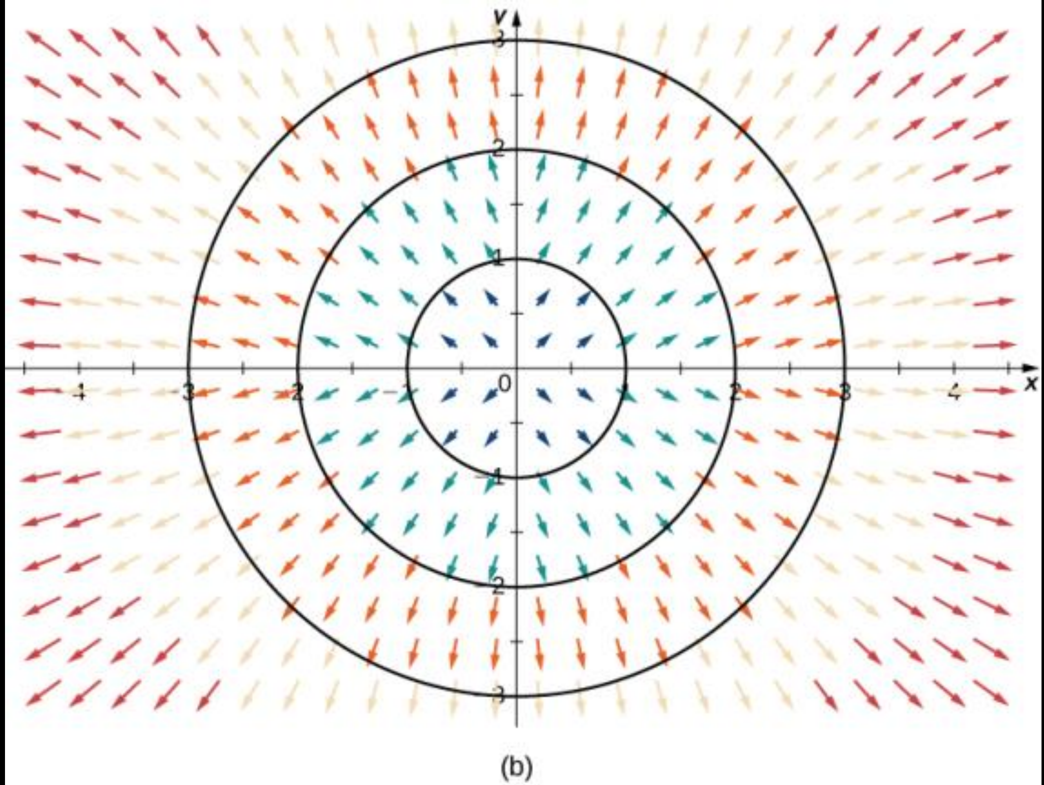
Escalares y vectores

Escalares -> Sólo magnitud

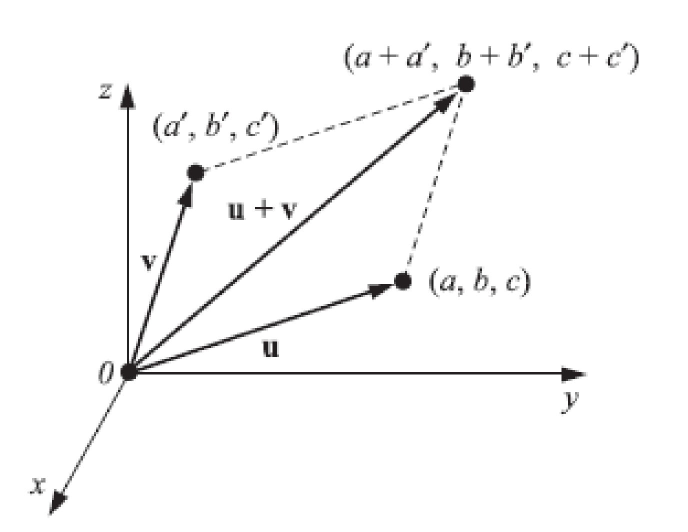


Vectores -> Magnitud, dirección y sentido

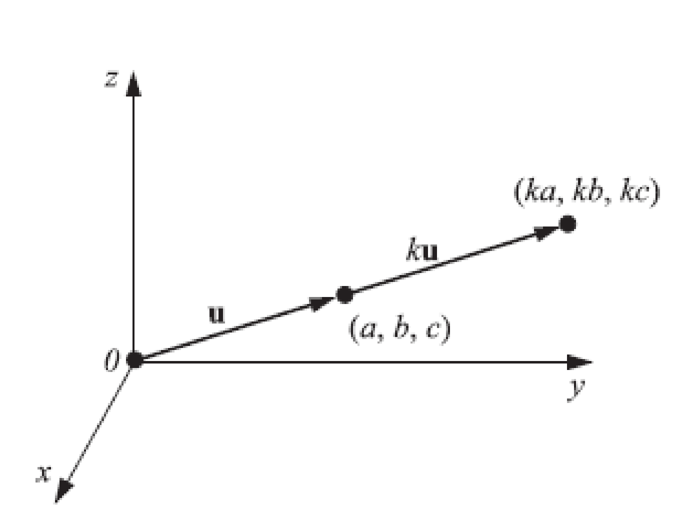


Many physical quantities, such as temperature and speed, possess only "magnitude." These quantities can be represented by real numbers and are called *scalars*. On the other hand, there are also quantities, such as force and velocity, that possess both "magnitude" and "direction." These quantities, which can be represented by arrows having appropriate lengths and directions and emanating from some given reference point O, are called *vectors*.

Vector Addition and Scalar Multiplication



(a) Vector Addition



(b) Scalar Multiplication

(i) ***Vector Addition:*** The resultant **u** + **v** of two vectors **u** and **v** is obtained by the *parallelogram law*; that is, **u** + **v** is the diagonal of the parallelogram formed by **u** and **v**. Furthermore, if (a, b, c) and (a', b', c') are the endpoints of the vectors **u** and **v**, then $(a+a',\ b+b',\ c+c')$ is the endpoint of the vector **u** + **v**. These properties are pictured in Fig. 1-1(a).

(ii) ***Scalar Multiplication:*** The product k**u** of a vector **u** by a real number k is obtained by multiplying the magnitude of **u** by k and retaining the same direction if $k > 0$ or the opposite direction if $k < 0$. Also, if (a, b, c) is the endpoint of the vector **u**, then (ka, kb, kc) is the endpoint of the vector k**u**. These properties are pictured in Fig. 1-1(b).

$$\bar{r} = \sum_{i=1}^{n} u_i + v_i$$

$$\bar{v} = \text{k}\,\bar{u}$$
$$\bar{v} = ku_1, ku_2, ku_3$$

Vector Addition and Scalar Multiplication

Consider two vectors u and v in $\mathbf{R}^n$, say

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

Their *sum*, written $u + v$, is the vector obtained by adding corresponding components from u and v. That is,

$$u + v = (a_1 + b_1,\ a_2 + b_2,\ \ldots,\ a_n + b_n)$$

The *scalar product* or, simply, *product*, of the vector u by a real number k, written ku, is the vector obtained by multiplying each component of u by k. That is,

$$ku = k(a_1, a_2, \ldots, a_n) = (ka_1, ka_2, \ldots, ka_n)$$

Now suppose we are given vectors $u_1, u_2, \ldots, u_m$ in $\mathbf{R}^n$ and scalars $k_1, k_2, \ldots, k_m$ in $\mathbf{R}$. We can multiply the vectors by the corresponding scalars and then add the resultant scalar products to form the vector

$$v = k_1u_1 + k_2u_2 + k_3u_3 + \cdots + k_mu_m$$

Such a vector v is called a *linear combination* of the vectors $u_1, u_2, \ldots, u_m$.

Let $u = (2, 4, -5)$ and $v = (1, -6, 9)$. Then

$$\begin{aligned} u + v &= (2+1,\ 4+(-5),\ -5+9) = (3, -1, 4) \\ 7u &= (7(2), 7(4), 7(-5)) = (14, 28, -35) \\ -v &= (-1)(1, -6, 9) = (-1, 6, -9) \\ 3u - 5v &= (6, 12, -15) + (-5, 30, -45) = (1, 42, -60) \end{aligned}$$

(c) Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $2u - 3v = \begin{bmatrix} 4 \\ 6 \\ -8 \end{bmatrix} + \begin{bmatrix} -9 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} -5 \\ 9 \\ -2 \end{bmatrix}$

Propiedades de vectores (adición y multiplicación por un escalar)

(i) $(u + v) + w = u + (v + w)$,

(ii) $u + 0 = u$,

(iii) $u + (-u) = 0$,

(iv) $u + v = v + u$,

(v) $k(u + v) = ku + kv$,

(vi) $(k + k')u = ku + k'u$,

(vii) $(kk')u = k(k'u)$,

(viii) $1u = u$.

Dot (Inner) Product

Consider arbitrary vectors u and v in $\mathbf{R}^n$; say,

$$u = (a_1, a_2, \ldots, a_n) \qquad \text{and} \qquad v = (b_1, b_2, \ldots, b_n)$$

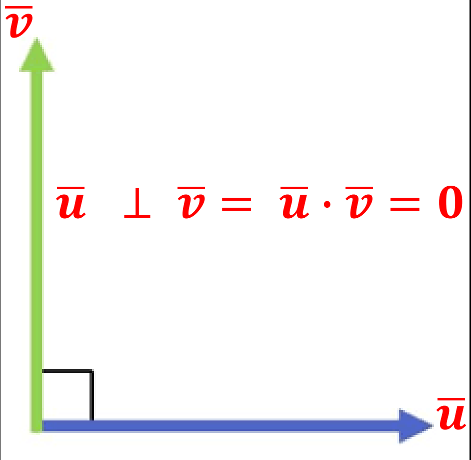
$$\# = \sum_{i=1}^{n} u_i v_i$$

The *dot product* or *inner product* or *scalar product* of u and v is denoted and defined by

$$u \cdot v = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

That is, $u \cdot v$ is obtained by multiplying corresponding components and adding the resulting products.

The vectors u and v are said to be *orthogonal* (or *perpendicular*) if their dot product is zero—that is, if $u \cdot v = 0$.



(i) $(u + v) \cdot w = u \cdot w + v \cdot w$, (iii) $u \cdot v = v \cdot u$,

(ii) $(ku) \cdot v = k(u \cdot v)$, (iv) $u \cdot u \geq 0$, and $u \cdot u = 0$ iff $u = 0$.

$$u \cdot (kv) = (kv) \cdot u = k(v \cdot u) = k(u \cdot v)$$

(a) Let $u = (1, -2, 3)$, $v = (4, 5, -1)$, $w = (2, 7, 4)$. Then,

$$u \cdot v = 1(4) - 2(5) + 3(-1) = 4 - 10 - 3 = -9$$
$$u \cdot w = 2 - 14 + 12 = 0, \qquad v \cdot w = 8 + 35 - 4 = 39$$

Thus, u and w are orthogonal.

(b) Let $u = \begin{bmatrix} 2 \\ 3 \\ -4 \end{bmatrix}$ and $v = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$. Then $u \cdot v = 6 - 3 + 8 = 11$.

(c) Suppose $u = (1, 2, 3, 4)$ and $v = (6, k, -8, 2)$. Find k so that u and v are orthogonal.

First obtain $u \cdot v = 6 + 2k - 24 + 8 = -10 + 2k$. Then set $u \cdot v = 0$ and solve for k:

$$-10 + 2k = 0 \qquad \text{or} \qquad 2k = 10 \qquad \text{or} \qquad k = 5$$

Let $u = (2, -7, 1)$, $v = (-3, 0, 4)$, $w = (0, 5, -8)$. Find:

(a) $3u - 4v$,

(b) $2u + 3v - 5w$.

Let $u = \begin{bmatrix} 5 \\ 3 \\ -4 \end{bmatrix}, v = \begin{bmatrix} -1 \\ 5 \\ 2 \end{bmatrix}, w = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$.

(a) $5u - 2v$,

(b) $-2u + 4v - 3w$.

Write the vector $v = (1, -2, 5)$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (2, -1, 1)$.

Find k so that u and v are orthogonal, where:

(a) $u = (1, k, -3)$ and $v = (2, -5, 4)$,

(b) $u = (2, 3k, -4, 1, 5)$ and $v = (6, -1, 3, 7, 2k)$.

Write the vector $v = (1, -2, 5)$ as a linear combination of the vectors $u_1 = (1, 1, 1)$, $u_2 = (1, 2, 3)$, $u_3 = (2, -1, 1)$.

We want to express v in the form $v = xu_1 + yu_2 + zu_3$ with x, y, z as yet unknown. First we have

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} x + y + 2z \\ x + 2y - z \\ x + 3y + z \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} x + y + 2z \\ x + 2y - z \\ x + 3y + z \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & -1 \\ 1 & 3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\mathbf{b} = \mathbf{A}\mathbf{x} \qquad A^{-1}[\,\mathbf{b} = \mathbf{A}\mathbf{x}\,] \qquad \mathbf{A}^{-1}\mathbf{b} = \mathbf{A}^{-1}\mathbf{A}\mathbf{x}$$

$$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{A}^{-1}\mathbf{b} = \mathbf{x}$$

Find k so that u and v are orthogonal, where:

(a) $u = (1, k, -3)$ and $v = (2, -5, 4)$,

(b) $u = (2, 3k, -4, 1, 5)$ and $v = (6, -1, 3, 7, 2k)$.

$$0 = \sum_{i=1}^{n} u_i v_i \qquad 0 = u_1 v_1 + k\, v_2 + u_3 v_3$$

Norm (Length) of a Vector

The *norm* or *length* of a vector u in $\mathbf{R}^n$, denoted by $\|u\|$, is defined to be the nonnegative square root of $u \cdot u$. In particular, if $u = (a_1, a_2, \ldots, a_n)$, then

$$\|u\| = \sqrt{u \cdot u} = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}$$

$$\|\boldsymbol{u}\| = \sqrt{\sum_{i=1}^{n} u_i u_i}$$

That is, $\|u\|$ is the square root of the sum of the squares of the components of u. Thus, $\|u\| \geq 0$, and $\|u\| = 0$ if and only if $u = 0$.

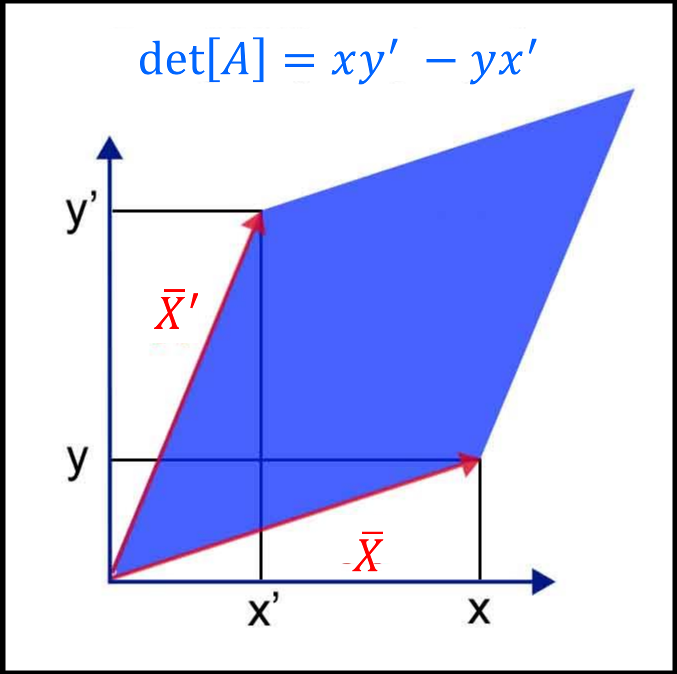
Determinantes

The **determinant** of the general 2 by 2 matrix $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as:

$$\det(\mathbf{A}) = ad - bc$$

a ↘ d minus c ↗ b

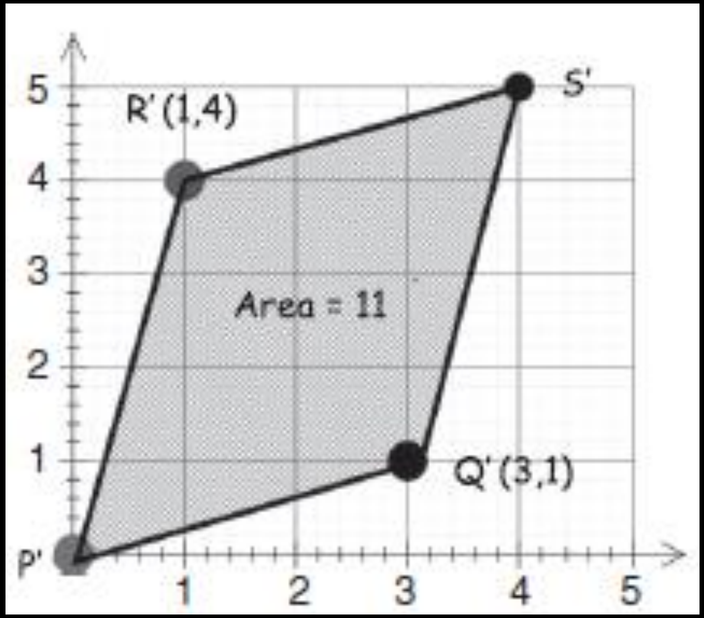
$\mathbf{A} = [\bar{X}, \bar{X}']$



$\bar{Q}$ $\bar{R}$

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$$

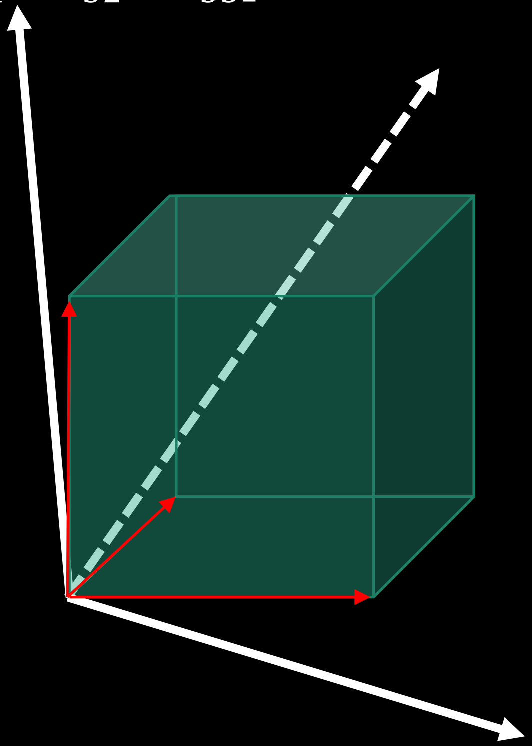
$\det[A] = 3(4) - 1(1) = 11$



Determinantes

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$\det[\mathbf{A}] = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$



El determinante de una matriz es igual a Cero si:

a) Dos renglones de la matriz son multiplos o iguales
b) Dos columnas de una matriz son multiplos o iguales
c) Un renglón o una columna son Cero (si la matriz no es cuadrada)

Si los vectores que forman la matriz no son linealmente independientes, det [A] = 0

La expansion de Laplace para el determinante de una matriz $\mathbf{A}$ de (nxn)

$$\det[\mathbf{A}] = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \forall \; i$$

M_{ij} es el determinante de la matriz (n-1) x (n-1) que resulta de remover el renglón i y la columna j de la matriz $\mathbf{A}$

$(-1)^{i+j} M_{ij}$ es el cofactor para el elemento a_{ij} de la matriz $\mathbf{A}$

La expansion de Laplace para el determinante de una matriz **A** de (nxn)

$$\det[\mathbf{A}] = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \forall \ i$$

M_{ij} es el determinante de la matriz (n-1) x (n-1) que resulta de remover el renglón i y la columna j de la matriz **A**

$(-1)^{i+j} M_{ij}$ es el cofactor para el elemento a_{ij} de la matriz **A**

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

$$A_{11} = +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18, \quad A_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2, \quad A_{13} = +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$$

$$A_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11, \quad A_{22} = +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14, \quad A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5$$

$$A_{31} = +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10, \quad A_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -4, \quad A_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$$

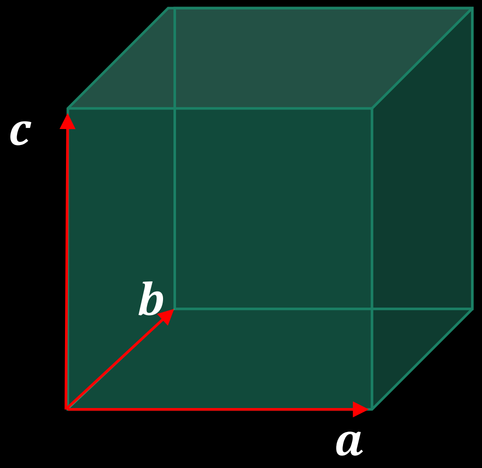
$$\det[\mathbf{A}] = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij} \quad \forall \ i \Rightarrow \det[A] = 2(-18) + 3(2) + (-4)(4)$$

$$+0(-11) + (-4)(14) + 2\,(5)$$

$$+1(-10) + (-1)(-4) + 5(-8)$$

$$\det(A) = -40 + 6 + 0 - 16 + 4 + 0 = -46$$

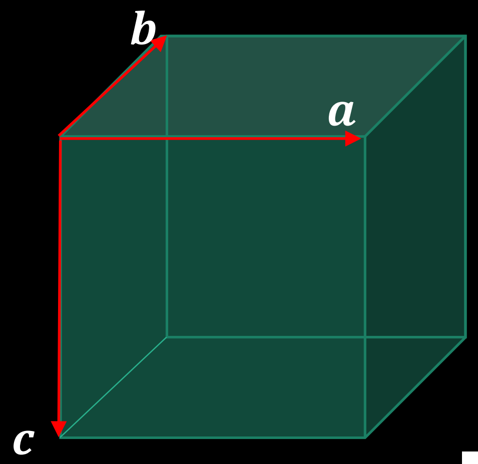
Cross Product



$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \sum_{j=1}^{n} (-1)^{i+j} ent_{ij} \mathbf{M}_{ij} \quad \forall \ i$$

ent_{ij} : *entrada ij*

$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\hat{\imath} - (a_1 b_3 - a_3 b_1)\hat{\jmath} + (a_1 b_2 - a_2 b_1)\hat{\boldsymbol{k}} = \boldsymbol{c}$$



¿El producto vectorial está definido solo para 3D?

$$\boldsymbol{b} \times \boldsymbol{a} = \begin{bmatrix} \hat{\imath} & \hat{\jmath} & \hat{\boldsymbol{k}} \\ b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \end{bmatrix} = \sum_{j=1}^{n} (-1)^{i+j} ent_{ij} \mathbf{M}_{ij} \quad \forall \ i$$

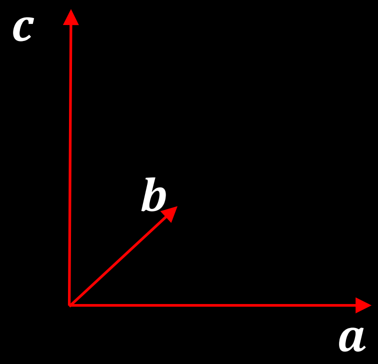
$$\boldsymbol{b} \times \boldsymbol{a} = (b_2 a_3 - b_3 a_2)\hat{\imath} - (b_1 a_3 - b_3 a_1)\hat{\jmath} + (b_1 a_2 - b_2 a_1)\hat{\boldsymbol{k}} = -\boldsymbol{c}$$

Remark: The cross products of the vectors $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are as follows:

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \qquad \mathbf{j} \times \mathbf{k} = \mathbf{i}, \qquad \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

$$\mathbf{j} \times \mathbf{i} = -\mathbf{k}, \qquad \mathbf{k} \times \mathbf{j} = -\mathbf{i}, \qquad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$$

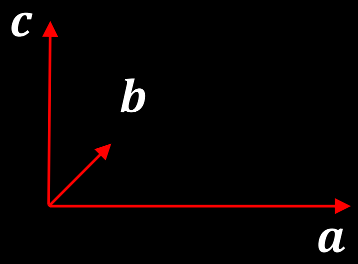
Cross Product



$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \widehat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \sum_{j=1}^{n} (-1)^{i+j} ent_{ij} \mathbf{M}_{ij} \quad \forall \ i$$

ent_{ij} : $entrada\ ij$

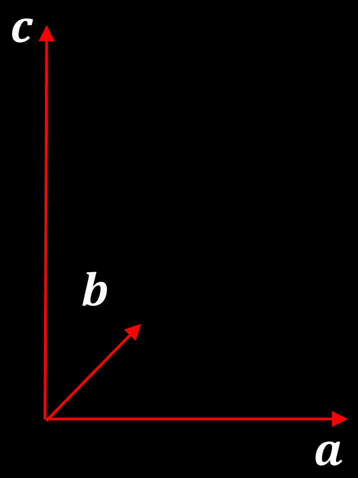
$$\boldsymbol{a} \times \boldsymbol{b} = (a_2 b_3 - a_3 b_2)\hat{\boldsymbol{\imath}} - (a_1 b_3 - a_3 b_1)\hat{\boldsymbol{\jmath}} + (a_1 b_2 - a_2 b_1)\widehat{\boldsymbol{k}} = \boldsymbol{c}$$



La magnitud del vector $\boldsymbol{c} = \boldsymbol{a} \times \boldsymbol{b}$

$$\|\boldsymbol{a} \times \boldsymbol{b}\| = \|\boldsymbol{a}\| \|\boldsymbol{b}\| \ sen\ \theta$$

θ es el ángulo entre los vectores $\boldsymbol{a}$ y $\boldsymbol{b}$



$$\|\boldsymbol{u}\| = \sqrt{\sum_{i=1}^{n} u_i u_i} \qquad \cos\theta = \frac{\sum_{i=1}^{n} u_i v_i}{\sqrt{\sum_{i=1}^{n} u_i u_i}\sqrt{\sum_{i=1}^{n} u_i u_i}}$$

Matrix vector vs vector matrix multiplication

$$\mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \end{pmatrix} \Rightarrow \mathbf{x}^T = (x_1 \quad x_2 \quad \dots)$$

El vector transpuesto de un vector columna es un vector renglón

El producto de una matriz por un vector $(\mathbf{A}\,\mathbf{x})$ *está definida si:*
el # de columnas de la matriz A es igual al # de renglones del vector columna

$\mathbf{A}\,\mathbf{x} \neq \mathbf{x}^T\mathbf{A}$

$\mathbf{A}\,\mathbf{x} = \mathbf{y}$

$$\mathbf{y} = \begin{pmatrix} a_{11} & \dots & a_{1m} \\ \vdots & \ddots & \ddots \\ a_{n1} & \dots & a_{nm} \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_m \end{pmatrix} = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \qquad y_i = \sum_{j=1}^{m} a_{ij} x_j \;\; \forall\, i = 1, 2, \dots, n$$

n es el número de renglones de A

El producto de un vector por una matriz $(\mathbf{x}^T\,\mathbf{A})$ *está definida si:*
el # de columnas del vector $\mathbf{x}$ *es igual al # de renglones de la matriz A*

$\mathbf{x}^T\mathbf{A} = \mathbf{z}$

$$\mathbf{z} = (x_1 \quad \dots \quad x_m) \begin{pmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \ddots \\ a_{m1} & \dots & a_{mn} \end{pmatrix} = (z_i \quad \dots \quad z_n) \qquad z_i = \sum_{j=1}^{m} x_j\, a_{ji} \;\; \forall\, i = 1, 2, \dots, n$$

n es el número de columnas de A

$\mathbf{A}\,\mathbf{x} = \mathbf{x}^T\mathbf{A}$ solo si A es simétrica

Matrices

A *matrix A over a field K* or, simply, a *matrix A* (when K is implicit) is a rectangular array of scalars usually presented in the following form:

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

The *rows* of such a matrix A are the m horizontal lists of scalars:

$$(a_{11}, a_{12}, \dots, a_{1n}), \quad (a_{21}, a_{22}, \dots, a_{2n}), \quad \dots, \quad (a_{m1}, a_{m2}, \dots, a_{mn})$$

and the *columns* of A are the n vertical lists of scalars:

$$\begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{m1} \end{bmatrix}, \quad \begin{bmatrix} a_{12} \\ a_{22} \\ \dots \\ a_{m2} \end{bmatrix}, \quad \dots, \quad \begin{bmatrix} a_{1n} \\ a_{2n} \\ \dots \\ a_{mn} \end{bmatrix}$$

Matrix Addition and Scalar Multiplication

Let $A = [a_{ij}]$ and $B = [b_{ij}]$ be two matrices with the same size, say $m \times n$ matrices.

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{bmatrix}$$

The *product* of the matrix A by a scalar k, written $k \cdot A$ or simply kA, is the matrix obtained by multiplying each element of A by k. That is,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \dots & \dots & \dots & \dots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{bmatrix}$$

Observe that $A + B$ and kA are also $m \times n$ matrices. We also define

$$-A = (-1)A \qquad \text{and} \qquad A - B = A + (-B)$$

La suma de matrices de diferente tamaño NO ESTÁ DEFINIDA

Reglas de la adición (sustracción) de matrices del mismo tamaño y de la multiplicación de estas por un escalar

Consider any matrices A, B, C (with the same size) and any scalars k and k'. Then

(i) $(A+B)+C = A+(B+C)$,

(ii) $A+0 = 0+A = A$,

(iii) $A+(-A) = (-A)+A = 0$,

(iv) $A+B = B+A$,

(v) $k(A+B) = kA+kB$,

(vi) $(k+k')A = kA+k'A$,

(vii) $(kk')A = k(k'A)$,

(viii) $1 \cdot A = A$.

Matrix Multiplication

The product AB of a row matrix $A = [a_i]$ and a column matrix $B = [b_i]$ with the same number of elements is defined to be the scalar (or 1×1 matrix) obtained by multiplying corresponding entries and adding; that is,

$$AB = [a_1, a_2, \ldots, a_n] \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_n \end{bmatrix} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = \sum_{k=1}^{n} a_k b_k$$

El # de columnas de A debe ser igual al # de renglones de B
El resultado de AB es un escalar

(a) $[7, -4, 5] \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = 7(3) + (-4)(2) + 5(-1) = 21 - 8 - 5 = 8$

(b) $[6, -1, 8, 3] \begin{bmatrix} 4 \\ -9 \\ -2 \\ 5 \end{bmatrix} = 24 + 9 - 16 + 15 = 32$

Matrix Multiplication

Suppose $A = [a_{ik}]$ and $B = [b_{kj}]$ are matrices such that the number of columns of A is equal to the number of rows of B; say, A is an $m \times p$ matrix and B is a $p \times n$ matrix. Then the product AB is the $m \times n$ matrix whose ij-entry is obtained by multiplying the ith row of A by the jth column of B. That is,

A B = C

$$\begin{bmatrix} a_{11} & \dots & a_{1p} \\ \cdot & \dots & \cdot \\ a_{i1} & \dots & a_{ip} \\ \cdot & \dots & \cdot \\ a_{m1} & \dots & a_{mp} \end{bmatrix} \begin{bmatrix} b_{11} & \dots & b_{1j} & \dots & b_{1n} \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ \cdot & \dots & \cdot & \dots & \cdot \\ b_{p1} & \dots & b_{pj} & \dots & b_{pn} \end{bmatrix} = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \cdot & \dots & \cdot \\ \cdot & c_{ij} & \cdot \\ \cdot & \dots & \cdot \\ c_{m1} & \dots & c_{mn} \end{bmatrix}$$

El # de columnas de A debe ser igual al # de renglones de B

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{ip}b_{pj} = \sum_{k=1}^{p} a_{ik}b_{kj}$$

La matriz C será de tamaño (m x n)
Con m como el # de renglones de A
Y n como el # de columnas de B

The product AB is not defined if A is an $m \times p$ matrix and B is a $q \times n$ matrix, where $p \neq q$.

Matrix Multiplication

Find AB where $A = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 0 & -4 \\ 5 & -2 & 6 \end{bmatrix}$

$$AB = \begin{bmatrix} 17 & -6 & 14 \\ 4-5 & 0+2 & -8-6 \end{bmatrix} = \begin{bmatrix} 17 & -6 & 14 \\ -1 & 2 & -14 \end{bmatrix}$$

Let A, B, C be matrices. Then, whenever the products and sums are defined,

(i) $(AB)C = A(BC)$ (associative law),

(ii) $A(B+C) = AB + AC$ (left distributive law),

(iii) $(B+C)A = BA + CA$ (right distributive law),

(iv) $k(AB) = (kA)B = A(kB)$, where k is a scalar.

$$A\,B \neq B\,A$$

A B = B A solo si A y B son simétricas

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 1 & 4 \end{pmatrix}$$

Transpose of a Matrix

The *transpose* of a matrix A, written A^T, is the matrix obtained by writing the columns of A, in order, as rows. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad [1, -3, -5]^T = \begin{bmatrix} 1 \\ -3 \\ -5 \end{bmatrix}$$

In other words, if $A = [a_{ij}]$ is an $m \times n$ matrix, then $A^T = [b_{ij}]$ is the $n \times m$ matrix where $b_{ij} = a_{ji}$.

Observe that the tranpose of a row vector is a column vector. Similarly, the transpose of a column vector is a row vector.

Let A and B be matrices and let k be a scalar. Then, whenever the sum and product are defined,

(i) $(A + B)^T = A^T + B^T$,

(ii) $(A^T)^T = A$,

(iii) $(kA)^T = kA^T$,

(iv) $(AB)^T = B^T A^T$.

Poner atención en (iv) : la transpuesta del producto de dos matrices es igual al producto de las matrices transpuestas llevando a cabo la multiplicación en el orden inverso

Diagonal and Trace

Let $A = [a_{ij}]$ be an n-square matrix. The *diagonal* or *main diagonal* of A consists of the elements with the same subscripts—that is,

$$a_{11}, \quad a_{22}, \quad a_{33}, \quad \ldots, \quad a_{nn}$$

The *trace* of A, written $\operatorname{tr}(A)$, is the sum of the diagonal elements. Namely,

$$\operatorname{tr}(A) = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}$$

Suppose $A = [a_{ij}]$ and $B = [b_{ij}]$ are n-square matrices and k is a scalar. Then

(i) $\operatorname{tr}(A + B) = \operatorname{tr}(A) + \operatorname{tr}(B)$, (iii) $\operatorname{tr}(A^T) = \operatorname{tr}(A)$,

(ii) $\operatorname{tr}(kA) = k \operatorname{tr}(A)$, (iv) $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

Powers of Matrices, Polynomials in Matrices

Let A be an n-square matrix over a field K. *Powers* of A are defined as follows:

$$A^2 = AA, \qquad A^3 = A^2A, \qquad \ldots, \qquad A^{n+1} = A^nA, \qquad \ldots, \qquad \text{and} \qquad A^0 = I$$

Polynomials in the matrix A are also defined. Specifically, for any polynomial

$$f(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

where the a_i are scalars in K, $f(A)$ is defined to be the following matrix:

$$f(A) = a_0I + a_1A + a_2A^2 + \cdots + a_nA^n$$

Inverse Matrix

Let A be any square matrix. Then

$$A(\text{adj } A) = (\text{adj } A)A = |A|I$$

$(\text{adj } A)$ is the transpose of the matrix of cofactors of A

where I is the identity matrix. Thus, if $|A| \neq 0$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$A\,(adj\ A) = |A|\ I$$

$$\frac{A\,(adj\ A)}{|A|} = I$$

$$A^{-1}\left(\frac{A\,(adj\ A)}{|A|}\right) = A^{-1}I$$

$$\frac{(adj\ A)}{|A|} = A^{-1}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$(\text{adj } A)$, is the transpose of the matrix of cofactors of A

$(-1)^{i+j} M_{ij}$ es el cofactor para el elemento a_{ij} de la matriz **A**

$$Cof(A) = \begin{pmatrix} +\begin{vmatrix} e & f \\ h & i \end{vmatrix} & -\begin{vmatrix} d & f \\ g & i \end{vmatrix} & +\begin{vmatrix} d & e \\ g & h \end{vmatrix} \\ -\begin{vmatrix} b & c \\ h & i \end{vmatrix} & +\begin{vmatrix} a & c \\ g & i \end{vmatrix} & -\begin{vmatrix} a & b \\ g & h \end{vmatrix} \\ +\begin{vmatrix} b & c \\ e & f \end{vmatrix} & -\begin{vmatrix} a & c \\ d & f \end{vmatrix} & +\begin{vmatrix} a & b \\ d & e \end{vmatrix} \end{pmatrix}$$

$$Adj(A) = \begin{pmatrix} ei - hf & -bi + hc & bf - ce \\ -di + gf & ai - gc & -af + cd \\ dh - ge & -ah + bg & ae - bd \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A)$$

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

$$A^{-1} = \frac{1}{aei + cdh + bfg - ceg - ahf - bdi} \begin{pmatrix} ei - hf & -bi + hc & bf - ce \\ -di + gf & ai - gc & -af + cd \\ dh - ge & -ah + bg & ae - bd \end{pmatrix}$$

$$A = \begin{bmatrix} 2 & 3 & -4 \\ 0 & -4 & 2 \\ 1 & -1 & 5 \end{bmatrix}$$

The cofactors of the nine elements of A follow:

$$A_{11} = +\begin{vmatrix} -4 & 2 \\ -1 & 5 \end{vmatrix} = -18, \quad A_{12} = -\begin{vmatrix} 0 & 2 \\ 1 & 5 \end{vmatrix} = 2, \quad A_{13} = +\begin{vmatrix} 0 & -4 \\ 1 & -1 \end{vmatrix} = 4$$

$$A_{21} = -\begin{vmatrix} 3 & -4 \\ -1 & 5 \end{vmatrix} = -11, \quad A_{22} = +\begin{vmatrix} 2 & -4 \\ 1 & 5 \end{vmatrix} = 14, \quad A_{23} = -\begin{vmatrix} 2 & 3 \\ 1 & -1 \end{vmatrix} = 5$$

$$A_{31} = +\begin{vmatrix} 3 & -4 \\ -4 & 2 \end{vmatrix} = -10, \quad A_{32} = -\begin{vmatrix} 2 & -4 \\ 0 & 2 \end{vmatrix} = -4, \quad A_{33} = +\begin{vmatrix} 2 & 3 \\ 0 & -4 \end{vmatrix} = -8$$

$$\text{adj } A = \begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix}$$

$$\det(A) = -40 + 6 + 0 - 16 + 4 + 0 = -46$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A) = -\frac{1}{46}\begin{bmatrix} -18 & -11 & -10 \\ 2 & 14 & -4 \\ 4 & 5 & -8 \end{bmatrix} = \begin{bmatrix} \frac{9}{23} & \frac{11}{46} & \frac{5}{23} \\ -\frac{1}{23} & -\frac{7}{23} & \frac{2}{23} \\ -\frac{2}{23} & -\frac{5}{46} & \frac{4}{23} \end{bmatrix}$$

Eigenvalues and Eigenvectors

Definition: A scalar λ is called an *eigenvalue* of the $n \times n$ matrix A is there is a nontrivial solution x of $A\mathrm{x} = \lambda\mathrm{x}$. Such an $\mathbf{x}$ is called an eigenvector *corresponding* to the eigenvalue λ.

$$
\begin{aligned}
A\mathrm{x} &= \lambda\mathrm{x} \\
A\mathrm{x} - \lambda\mathrm{x} &= \mathbf{0} \\
A\mathrm{x} - \lambda I\mathrm{x} &= \mathbf{0} \\
(A - \lambda I)\mathrm{x} &= \mathbf{0}
\end{aligned}
$$

The Characteristic Equation

A scalar λ is an eigenvalue of an $n \times n$ matrix A if and only if λ satisfies the *characteristic equation*

$$\det(A - \lambda I) = 0.$$

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

$$\mathbf{A} - \lambda\mathbf{I} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$= \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix}$$

$$\det(\mathbf{A} - \lambda\mathbf{I}) = \det \begin{pmatrix} 2-\lambda & 0 \\ 1 & 3-\lambda \end{pmatrix} = (2-\lambda)(3-\lambda) - 0$$

$(2-\lambda)(3-\lambda) = 0$ implies $\lambda_1 = 2$ or $\lambda_2 = 3$

Eigenvectors

For each eigenvalue, λ determine the corresponding eigenvector $\mathbf{u}$ by solving the system $(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}$.

$$\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix}$$

$(2-\lambda)(3-\lambda) = 0$ implies $\lambda_1 = 2$ or $\lambda_2 = 3$

$\lambda_1 = \lambda = 2$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\right]\mathbf{u} = \mathbf{O}$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\mathbf{u} = \mathbf{O}$$

$\mathbf{u} = \begin{pmatrix} x \\ y \end{pmatrix}$, so we have

$$\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$0 + 0 = 0$$
$$x + y = 0$$

From the bottom equation we have $x = -y$.

The simplest solution is $x = 1$, $y = -1$ but we could have $\begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -3 \\ 3 \end{pmatrix}, \begin{pmatrix} \pi \\ -\pi \end{pmatrix}, \begin{pmatrix} 5 \\ -5 \end{pmatrix}, \ldots$

$\lambda_2 = 3$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{v} = \left[\begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} - \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}\right]\mathbf{v} = \mathbf{O}$$

$$\begin{pmatrix} -1 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-x + 0 = 0, \quad x + 0 = 0$$

From these equations we must have $x = 0$

$y = s$ where $s \neq 0$

$$A = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3 & 6 & -8 \\ 0 & 0 & 6 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = \begin{bmatrix} 3-\lambda & 6 & -8 \\ 0 & -\lambda & 6 \\ 0 & 0 & 2-\lambda \end{bmatrix}$$

$$\begin{aligned} \det(A - \lambda I) &= (3-\lambda)(-\lambda)(2-\lambda) + (6)(6)(0) + (-8)(0)(0) \\ &\quad - (0)(-\lambda)(-8) - (0)(6)(3-\lambda) - (-\lambda)(0)(6) \\ &= -\lambda(3-\lambda)(2-\lambda) \end{aligned}$$

Setting this equal to 0 and solving for λ, we get that $\lambda = 0$, 2, or 3. These are the three eigenvalues of A.

$$A = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix}$$

$$(\mathbf{A} - \lambda\mathbf{I})\mathbf{u} = \mathbf{O}.$$

$$|A| = -4 - 6 = -10$$

$$MX = 0$$

$$\det(A - tI) = t^2 - 3t - 10 = (t-5)(t+2)$$

The roots $\lambda_1 = 5$ and $\lambda_2 = -2$ are the eigenvalues of A.

The eigenvectors belonging to $\lambda_1 = 5$

$$\begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} -x + 2y &= 0 \\ 3x - 6y &= 0 \end{aligned} \quad \text{or} \quad -x + 2y = 0$$

$$v_1 = (2, 1)$$

$$\lambda_2 = -2$$

$$\begin{aligned} 6x + 2y &= 0 \\ 3x + y &= 0 \end{aligned} \quad \text{or} \quad 3x + y = 0$$

$$v_2 = (-1, 3)$$

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{bmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \widehat{\boldsymbol{k}} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{bmatrix} = \sum_{j=1}^{n} (-1)^{i+j} ent_{ij} \mathbf{M}_{ij} \quad \forall \; i$$

D = a x b x c

=	i	j	k	l
	a1	a2	a3	a4
	b1	b2	b3	b4
	c1	c2	c3	c4

Aplicaciones de la derivada (Optimización)

Un problema de optimización consiste en minimizar o maximizar el valor de una variable. En otras palabras se trata de calcular o determinar el valor mínimo o el valor máximo de una función de una variable.

Se debe tener presente que la variable que se desea minimizar o maximizar debe ser expresada como función de otra de las variables relacionadas en el problema.

En ocasiones es preciso considerar las restricciones que se tengan en el problema, ya que éstas generan igualdades entre las variables que permiten la obtención de la función de una variable que se quiere minimizar o maximizar.

En este tipo de problemas se debe contestar correctamente las siguientes preguntas:

- ¿Qué se solicita en el problema?
- ¿Qué restricciones aparecen en el problema?

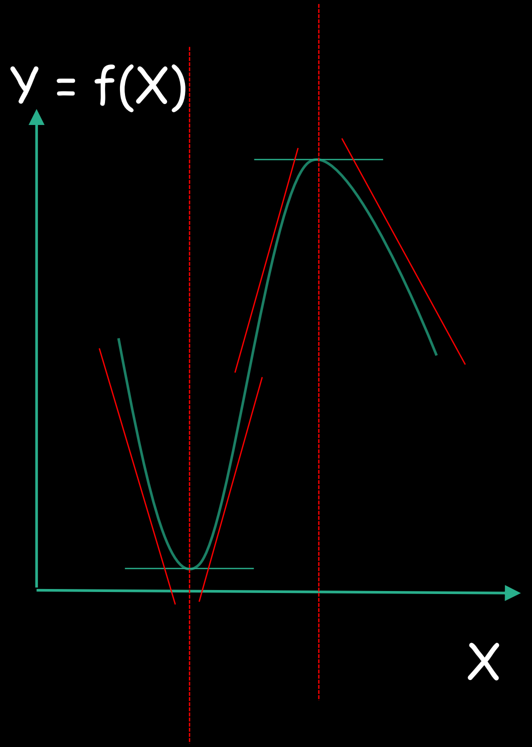
La respuesta correcta a la primera pregunta nos lleva a definir la función que deberá ser minimizada o maximizada.

La respuesta correcta a la segunda pregunta dará origen a (al menos) una ecuación que será auxiliar para lograr expresar a la función deseada precisamente como una función de una variable.

Hipótesis. Si $f(x)$ es una función continua en el intervalo (a,b), x_1 es el único punto crítico en ese intervalo y $f(x)$ es derivable en (a,b), entonces:

Tesis. El punto crítico $(x_1, f(x_1))$ se puede clasificar de acuerdo con la siguiente tabla:

Signo de $f'(x)$ en (a,x_1)	Signo de $f'(x)$ en (x_1,b)	Decisión
+	-	$f(x_1)$ es un máximo
-	+	$f(x_1)$ es un mínimo

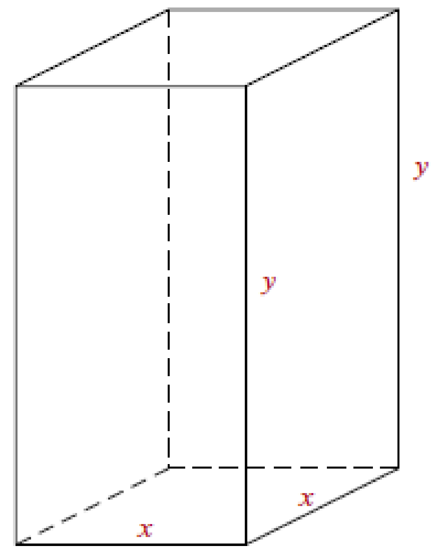


Hipótesis. Si $f(x)$ es una función que tiene primera y segunda derivada en un intervalo (a,b) y tiene un punto crítico en $x = x_1$.

Tesis. El punto crítico $(x_1, f(x_1))$ puede clasificarse de acuerdo con el siguiente criterio:

- Si $f''(x_1) > 0$, $f(x)$ tendrá un mínimo en x_1, porque es cóncava hacia arriba.
- Si $f''(x_1) < 0$, $f(x)$ tendrá un máximo en x_1, porque es cóncava hacia abajo.

Ejemplo 10.1.1 *Una caja con base cuadrada y parte superior abierta debe tener un volumen de 50 cm^3. Encuentre las dimensiones de la caja que minimicen la cantidad de material que va a ser usado.*



$V = x^2 y$ & $V = 50 \Rightarrow$

$50 = x^2 y$; esta igualdad relaciona las variables del problema.

El área de la caja sin tapa:

$$A = x^2 + 4xy\,.$$

Ésta es la cantidad de material que deseamos que sea mínima; vemos que es una función de dos variables.

Despejamos y de la restricción dada, esto es, de la fórmula del volumen:

$$y = \frac{50}{x^2}\,.$$

Sustituimos en el área y obtenemos una función de una sola variable:

$$A(x) = x^2 + 4x\left(\frac{50}{x^2}\right) = x^2 + \frac{200}{x} = x^2 + 200x^{-1}.$$

$$f(u) = cx^n \rightarrow f'(x) = c\,n\,x^{n-1}x'$$

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{vu' - uv'}{v^2}$$

$$A'(x) = 2x - 200x^{-2} = 2x - \frac{200}{x^2} = \frac{2x^3 - 200}{x^2}$$

$$A'(x) = \frac{2x^3 - 200}{x^2} = 0$$

Calculamos puntos críticos:

$$A'(x) = 0 \Rightarrow 2x^3 - 200 = 0 \Rightarrow x^3 = 100 \Rightarrow x = \sqrt[3]{100}$$

$V = x^2y$ & $V = 50 \Rightarrow$

$50 = x^2y$; esta igualdad relaciona las variables del problema.

$$x = \sqrt[3]{100} = 100^{1/3}$$

$$x^2 = \left(100^{1/3}\right)^2$$

$$y = \frac{50}{100^{\frac{2}{3}}}$$

$$A''(x) = 2 + 200\left(\frac{2}{x^3}\right) = 2 + \frac{400}{x^3} > 0$$

→ Minimizar f(x)

$$A'(x) = \frac{2x^3 - 200}{x^2} = 0$$

$$A'(x) = 2x - \frac{200}{x^2}$$

$$\frac{d(A'(x))}{dx} = \frac{d}{dx}(2x) - \frac{d}{dx}\left(\frac{200}{x^2}\right)$$

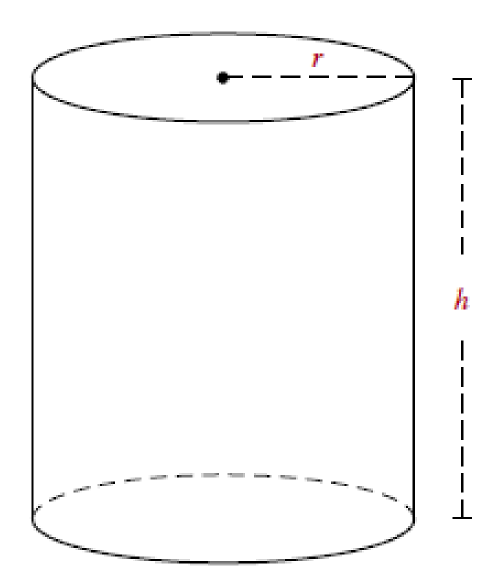
$$= 2\frac{d(x)}{dx} - \left[\frac{x^2 \frac{d\,200}{dx} - 200\frac{d(x^2)}{dx}}{(x^2)^2}\right]$$

$$= 2 - \left[\frac{0 - 200\left(2x\frac{dx}{dx}\right)}{x^4}\right]$$

$$= 2 - \left[\frac{-400x}{x^4}\right]$$

$$= 2 + \frac{400}{x^3}$$

Ejemplo 10.1.5 *Se desea construir un recipiente cilíndrico de metal con tapa que tenga una superficie total de 80 cm^2. Determine sus dimensiones de modo que tenga el mayor volumen posible.*



Se desea maximizar el volumen $V = \pi r^2 h$ que depende de dos variables r & h.
Se sabe que el área total $A = 2\pi r^2 + 2\pi r h$ debe ser igual a 80 cm^2.
Es decir, se sabe que $2\pi r^2 + 2\pi r h = 80$.

$$V = \pi r^2 h;$$

$$2\pi r^2 + 2\pi r h = 80$$

$$2\pi r^2 + 2\pi r h = 80 \Rightarrow \pi r^2 + \pi r h = 40 \Rightarrow \pi r h = 40 - \pi r^2 \Rightarrow h = \frac{40 - \pi r^2}{\pi r}$$

$V = \pi r^2 h = \pi r^2 \left(\frac{40 - \pi r^2}{\pi r}\right) = r(40 - \pi r^2) \Rightarrow V(r) = 40r - \pi r^3$ que es la función a maximizar.

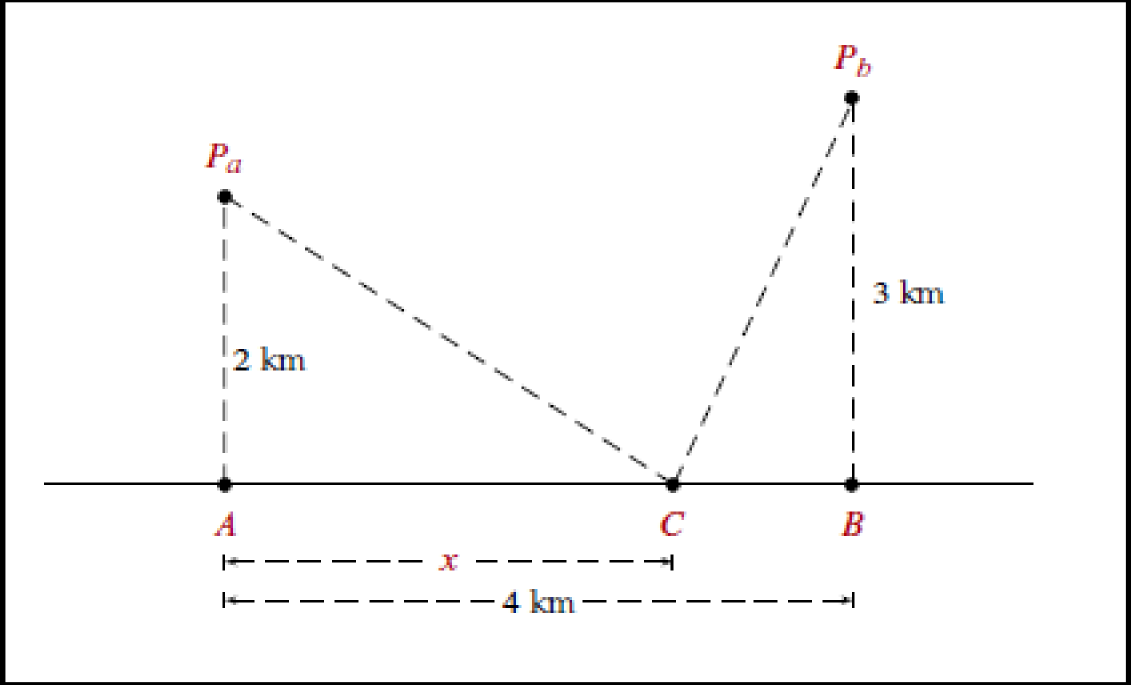
$$V'(r) = 40 - 3\pi r^2;$$

$$V'(r) = 0 \Leftrightarrow 40 - 3\pi r^2 = 0 \Leftrightarrow r^2 = \frac{40}{3\pi} \approx 4.2441$$

$$\Rightarrow r = \pm\sqrt{4.2441} \approx \pm 2.0601.$$

$$V''(r_1) = -6\pi r_1 \approx -6\pi(2.0601) < 0$$

Ejemplo 10.1.7 *Dos poblados P_a y P_b están a 2 km y 3 km, respectivamente, de los puntos más cercanos A y B sobre una línea de transmisión, los cuales están a 4 km uno del otro. Si los dos poblados se van a conectar con un cable a un mismo punto de la línea, ¿cuál debe ser la ubicación de dicho punto para utilizar el mínimo de cable?*



Sea C el punto de conexión ubicado, digamos, a x km del punto A y por supuesto a $4 - x$ km del punto B.
Si l es la longitud del cable utilizado para conectar P_a y P_b con C, entonces:

$$l = \overline{P_aC} + \overline{P_bC} = \sqrt{x^2 + 2^2} + \sqrt{(4 - x)^2 + 3^3}\,.$$

La función a minimizar es:

$$l(x) = \sqrt{x^2 + 4} + \sqrt{(4 - x)^2 + 9} = (x^2 + 4)^{\frac{1}{2}} + \left[(4 - x)^2 + 9\right]^{\frac{1}{2}}$$

Derivando y obteniendo puntos críticos:

$$l'(x) = \frac{1}{2}(x^2 + 4)^{-\frac{1}{2}}2x + \frac{1}{2}\left[(4 - x)^2 + 9\right]^{-\frac{1}{2}}2(4 - x)(-1)\,;$$

$$l'(x) = \frac{x}{\sqrt{x^2 + 4}} - \frac{4 - x}{\sqrt{(4 - x)^2 + 9}}\,;$$

$$l'(x) = 0 \Rightarrow \frac{x}{\sqrt{x^2 + 4}} - \frac{4 - x}{\sqrt{(4 - x)^2 + 9}} = 0 \Rightarrow \frac{x}{\sqrt{x^2 + 4}} = \frac{4 - x}{\sqrt{(4 - x)^2 + 9}}$$

$$\Rightarrow x\sqrt{(4 - x)^2 + 9} = (4 - x)\sqrt{x^2 + 4}\,.$$

Elevando al cuadrado ambos miembros de la igualdad:

$$x^2[(4-x)^2+9] = (4-x)^2(x^2+4) \Rightarrow x^2(4-x)^2 + 9x^2 = x^2(4-x)^2 + 4(4-x)^2 \Rightarrow$$
$$\Rightarrow 9x^2 = 4(4-x)^2 \Rightarrow 9x^2 = 4(16-8x+x^2) \Rightarrow$$
$$\Rightarrow 9x^2 = 64 - 32x + 4x^2 \Rightarrow 5x^2 + 32x - 64 = 0\,.$$

Esta última ecuación tiene por soluciones:

$$x = \frac{-32 \pm \sqrt{(32)^2 - 4(5)(-64)}}{2(5)} = \frac{-32 \pm \sqrt{2304}}{10} = \frac{-32 \pm 48}{10}.$$

De donde se obtienen dos puntos críticos que son:

$$x_1 = \frac{-32+48}{10} = \frac{16}{10} = 1.6 \text{ así como } x_2 = \frac{-32-48}{10} = \frac{-80}{10} = -8\,.$$

Claramente el valor $x_2 = -8 < 0$ es descartado y sólo consideramos $x_1 = 1.6$.

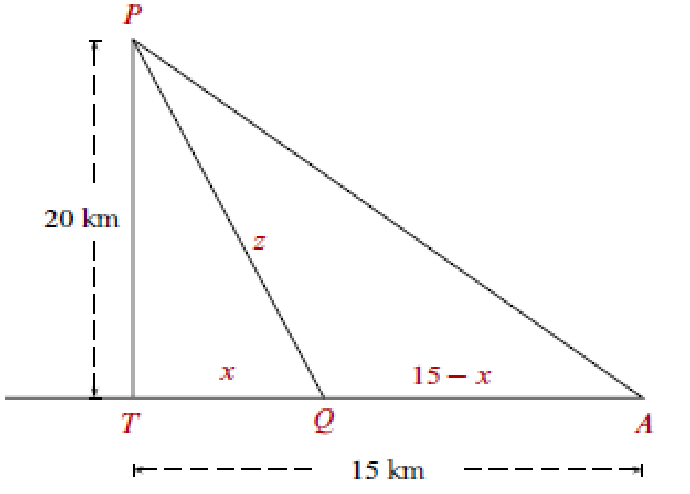
entonces $l''(x) > 0$ para cada x. En particular $l''(1.6) > 0$, por lo que $l(x)$ es mínima cuando $x = 1.6$ km.

Puesto que $0 \leq x \leq 4$, calculemos los números $l(0)$, $l(1.6)$ y $l(4)$ a manera de ejemplo:

$$l(0) = \sqrt{0^2 + 4} + \sqrt{(4-0)^2 + 9} = 2 + \sqrt{25} = 2 + 5 = 7\,;$$
$$l(1.6) = \sqrt{(1.6)^2 + 4} + \sqrt{(4-1.6)^2 + 9} = \sqrt{6.56} + \sqrt{14.76} \approx 6.4\,;$$
$$l(4) = \sqrt{4^2 + 4} + \sqrt{(4-4)^2 + 9} = \sqrt{20} + 3 \approx 7.5\,.$$

Se ve pues que $l(x)$ es menor cuando $x = 1.6$ km, siendo la longitud mínima del cable igual a 6.4 km aproximadamente.

Ejemplo 10.1.8 *Se requiere construir un oleoducto desde una plataforma marina que está localizada al norte 20 km mar adentro, hasta unos tanques de almacenamiento que están en la playa y 15 km al este. Si el costo de construcción de cada km de oleoducto en el mar es de 2 000 000 de dólares y en tierra es de 1 000 000, ¿a qué distancia hacia el este debe salir el oleoducto submarino a la playa para que el costo de la construcción sea mínimo?*

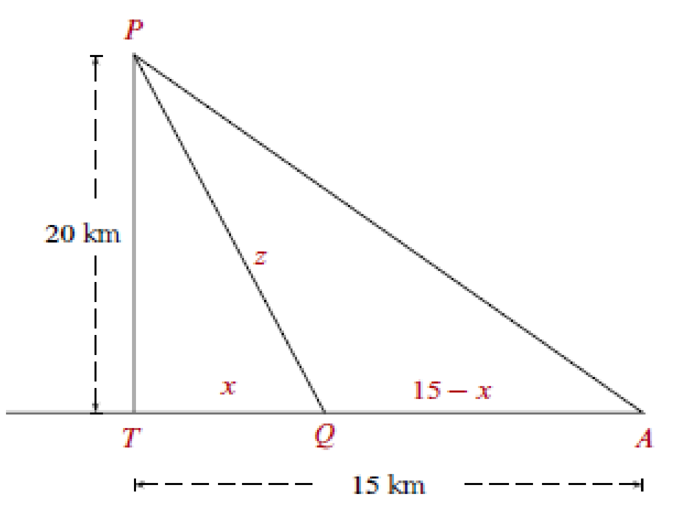


Consideremos que la plataforma está en P y que T es el punto de la playa más cercano a ella; que A es donde están los tanques de almacenamiento y Q es el punto de la playa donde debe de salir el oleoducto submarino.

Si x representa la distancia del punto T al punto Q, entonces considerando el triángulo rectángulo PTQ:

$$z^2 = x^2 + (20)^2 \Rightarrow z = \sqrt{x^2 + 400},$$

que es la porción de oleoducto submarino; $\overline{QA} = 15 - x$ es la porción de oleoducto en tierra.



$0 \leq x \leq 15$

El costo de construir z kilómetros de oleoducto submarino, a razón de 2 000 000 de dólares por km es de $2z$ millones de dólares y el costo de construir $15-x$ km de oleoducto terrestre, a razón de 1 000 000 de dólares por km es $1 \cdot (15-x)$ millones de dólares. Entonces, el costo total de la construcción del oleoducto (en millones de dólares) es

$$C = 2z + (15-x) = 2\sqrt{x^2+400} + 15 - x \ \Rightarrow\ C(x) = 2\sqrt{x^2+400} + 15 - x = 2(x^2+400)^{\frac{1}{2}} + 15 - x$$

$$C'(x) = 2\left(\frac{1}{2}\right)(x^2+400)^{-\frac{1}{2}}2x - 1 = \frac{2x}{\sqrt{x^2+400}} - 1\,;$$

$$C'(x) = 0 \;\Rightarrow\; C'(x) = \frac{2x}{\sqrt{x^2-400}} - 1 = 0 \;\Rightarrow\; \frac{2x}{\sqrt{x^2+400}} = 1 \;\Rightarrow$$

$$\Rightarrow\; 2x = \sqrt{x^2+400} \;\Rightarrow\; 4x^2 = x^2 + 400 \;\Rightarrow\; 3x^2 = 400 \;\Rightarrow$$

$$\Rightarrow\; x^2 = \frac{400}{3} \;\Rightarrow\; x = \pm\frac{20}{\sqrt{3}}\,.$$

$$C''(x) = \frac{-2x^2 + 2(x^2 + 400)}{(x^2 + 400)^{\frac{3}{2}}} = \frac{800}{(x^2 + 400)^{\frac{3}{2}}}\,.$$

Vemos que $C''(x) > 0$ para cualquier x.

Por lo cual $C(x)$ tiene un mínimo local estricto en $x = \dfrac{20}{\sqrt{3}}$.

Ahora bien, no debemos olvidar que $0 \le x \le 15$.

¿Cuál será el costo $C(x)$ en los casos extremos $x = 0$ y en $x = 15$?

Ya que $C(x) = 2\sqrt{x^2 + 400} + 15 - x$, entonces

$$C(0) = 2\sqrt{400} + 15 = 55\,;$$

$$C\left(\frac{20}{\sqrt{3}}\right) = 2\sqrt{\frac{400}{9} + 400} + 15 - \frac{20}{\sqrt{3}} = 2\sqrt{\frac{400 + 3\,600}{9}} + 15 - \frac{20}{\sqrt{3}} = 2\sqrt{\frac{4\,000}{9}} + 15 - \frac{20}{\sqrt{3}} =$$

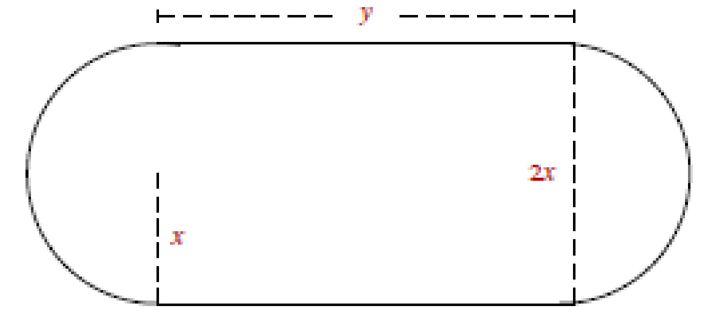
$$= \frac{2\sqrt{4\,000}}{3} + 15 - \frac{20}{\sqrt{3}} \approx 45.6167\,;$$

$$C(15) = 2\sqrt{225 + 400} + 15 - 15 = 50\,.$$

El costo mínimo de la construcción del oleoducto es de 45.6167 millones de dólares y se tiene cuando $x = \dfrac{20}{\sqrt{3}} \approx 11.547$ km.

□

Ejemplo 10.1.3 *Un terreno tiene la forma de un rectángulo con dos semicírculos en los extremos. Si el perímetro del terreno es de 50 m, encontrar las dimensiones del terreno para que tenga el área máxima.*



El área del terreno es

$$A = 2xy + \pi x^2 .$$

El perímetro, $P = 50$ m, está dado por $P = 2y + 2\pi x$, por lo que

$$2y + 2\pi x = 50 \Rightarrow y = \frac{50 - 2\pi x}{2} = 25 - \pi x.$$

Si sustituimos este valor en la fórmula del área, la tendremos expresada como función de una variable x:

$$A(x) = 2x(25 - \pi x) + \pi x^2 = 50x + x^2(\pi - 2\pi) = 50x - \pi x^2.$$

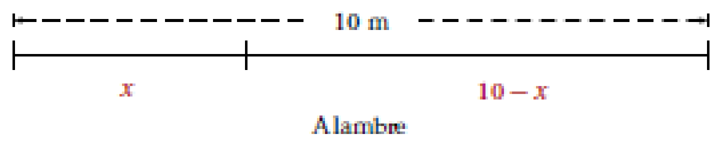
Su punto crítico se obtiene cuando $A'(x) = 0$. Esto es:

$$A'(x) = \left(50x - \pi x^2\right)' = 50 - 2\pi x = 0 \Leftrightarrow x = \frac{50}{2\pi} = \frac{25}{\pi} .$$

Como $A''(x) = -2\pi < 0$, se trata en efecto de un máximo; además $y = 25 - \pi\frac{25}{\pi} = 0$, es decir, el área máxima se obtiene cuando el terreno tiene la forma circular.

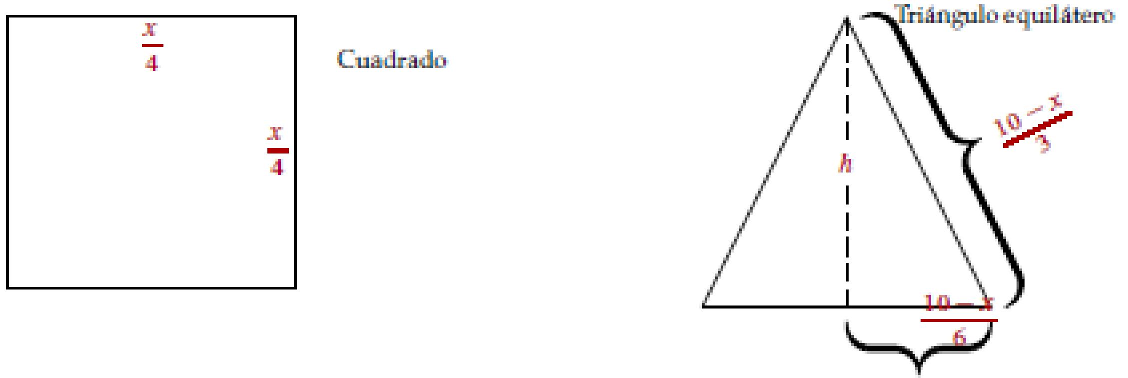
Ejemplo 10.1.11 *Un trozo de alambre de 10 m de largo se corta en dos partes. Una se dobla para formar un cuadrado y la otra para formar un triángulo equilátero. Hallar cómo debe cortarse el alambre de modo que el área encerrada sea:*

1. *Máxima.*
2. *Mínima.*



La parte x del alambre se usa para el cuadrado, por lo tanto cada lado tiene longitud $\frac{x}{4}$.

La parte $10-x$ del alambre se usa para el triángulo equilátero, por lo tanto cada lado tiene longitud $\frac{10-x}{3}$.



De la figura del triángulo, usando el teorema de Pitágoras, obtenemos la siguiente relación:

$$h^2 + \left(\frac{10-x}{6}\right)^2 = \left(\frac{10-x}{3}\right)^2 \Rightarrow h^2 = \frac{1}{9}(10-x)^2 - \frac{1}{36}(10-x)^2 \Rightarrow$$

$$\Rightarrow h^2 = \frac{3}{36}(10-x)^2 \Rightarrow h = \frac{\sqrt{3}}{6}(10-x).$$

El área del cuadrado es

$$A_C(x) = \frac{x^2}{16}.$$

El área del triángulo es

$$A_T(x) = \frac{1}{2} \times \text{base} \times \text{altura} = \frac{1}{2} \times \frac{1}{3}(10-x) \times \frac{\sqrt{3}}{6}(10-x) = \frac{\sqrt{3}}{36}(10-x)^2.$$

El área de ambas figuras es

$$A(x) = A_T(x) + A_C(x) = \frac{\sqrt{3}}{36}(10-x)^2 + \frac{x^2}{16}.$$

Ésta es la función a la cual deseamos calcular sus máximo y mínimo.

Nótese que el dominio de esta función es $D_A = [0, 10]$ (la longitud del alambre es de 10 m).

Calculamos la primera derivada:

$$A'(x) = \frac{1}{8}x + \frac{\sqrt{3}}{36} \times 2(10-x)(-1) = \frac{1}{8}x - \frac{\sqrt{3}}{18}(10-x) =$$
$$= \frac{1}{8}x - \frac{5\sqrt{3}}{9} + \frac{\sqrt{3}}{18}x = \frac{9+4\sqrt{3}}{72}x - \frac{5\sqrt{3}}{9}.$$

Calculamos el punto crítico:

$$A'(x) = 0 \Rightarrow \frac{9+4\sqrt{3}}{72}x - \frac{5\sqrt{3}}{9} = 0 \Rightarrow x = \frac{72(5\sqrt{3})}{9(9+4\sqrt{3})} = \frac{40\sqrt{3}}{9+4\sqrt{3}} \approx 4.34965.$$

Puesto que al calcular la segunda derivada obtenemos:

$$A''(x) = \frac{9 + 4\sqrt{3}}{72} > 0,$$

entonces el punto crítico anterior es un mínimo local.
Calculamos la función $A(x)$ en los extremos de su dominio:

$$A(0) = \frac{\sqrt{3}}{36}100 \approx 4.81125 \text{ y } A(10) = \frac{100}{16} = 6.25.$$

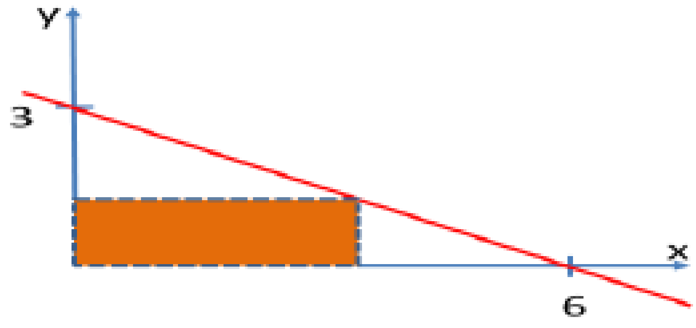
Vemos entonces que la máxima área encerrada es cuando $x = 10$, es decir, cuando sólo se construye el cuadrado.
Y la mínima área encerrada es cuando $x = 4.34965$, caso en el que se construyen ambas figuras.

2.- Un rectángulo está inscrito en el primer cuadrante ($x > 0$, $y > 0$) de un eje coordenado (x,y). La base se encuentra sobre la línea del eje x (y=0) y la altura sobre la línea del eje y (x=0). Si el vértice superior derecho del rectángulo toca un punto de la línea y = (6 - x)/2, cuáles son las dimensiones del rectángulo para que su área sea máxima.

Área = $A = x\,y$

Si $y = (6 - x)/2 \Rightarrow (x=6, y=0)$

$(x=0, y=3)$



Por lo tanto las dimensiones del rectángulo deben cumplir con:

$$0 < x < 6$$

$$0 < y < 3$$

$A = x(6 - x)/2 = 3x - 0.5x^2$

$A' = 3 - x$, $A' = 3 - x = 0$ => El punto crítico para A es $x = 3$

$A'' = -1$ => $x = 3$ es un máximo

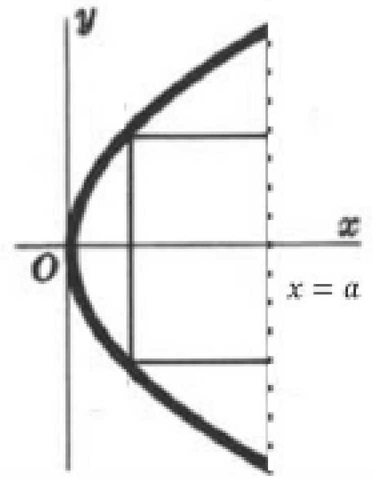
$y(3) = (6 - 3)/2 = 1.5$

Las dimensiones del rectángulo son:

$x = 3$

$y = 1.5$

Encuentra el rectángulo de área máxima A que puede ser inscrito en la porción de la parábola $y^2 = 4px$ intersectado por la línea punteada vertical x=a.



$$A = 2y(a - x) = 2y\left(a - \frac{y^2}{4p}\right) = 2ay - \frac{y^3}{2p} \qquad \text{and} \qquad \frac{dA}{dy} = 2a - \frac{3y^2}{2p}$$

Solving $dA/dy = 0$ yields the critical value $y = \sqrt{4ap/3}$.

$$a - x = a - y^2/4p = 2a/3.$$

Since $\frac{d^2A}{dy^2} = -\frac{3}{p}y < 0$, the second-derivative test and the uniqueness of the critical value ensure that we have found the maximum area.

Differential Equations

A *differential equation* is an equation involving an unknown function and its derivatives.

A differential equation is an *ordinary differential equation* if the unknown function depends on only one independent variable. If the unknown function depends on two or more independent variables, the differential equation is a *partial differential equation*.

The *order* of a differential equation is the order of the highest derivative appearing in the equation.

A *solution* of a differential equation in the unknown function y and the independent variable x on the interval $\mathcal{J}$ is a function $y(x)$ that satisfies the differential equation identically for all x in $\mathcal{J}$.

First-Order Differential Equations

- *Separable Equations*
- *Homogeneous Equations*
- *Exact Equations*
- *Linear Equations*
- *Bernoulli Equations*

Separable Equations

$$A(x)dx + B(y)dy = 0$$

$$\int A(x)dx + \int B(y)dy = c$$

Homogeneous Equations

The homogeneous differential equation

$$\frac{dy}{dx} = f(x, y)$$

can be transformed into a separable equation by making the substitution

$$y = xv$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

Exact Equations

$$M(x, y)dx + N(x, y)dy = 0$$

is *exact* if there exists a function $g(x, y)$ such that

$$dg(x, y) = M(x, y)dx + N(x, y)dy$$

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

Linear Equations

$$y' + p(x)y = q(x)$$

$$I(x) = e^{\int p(x)dx}$$

$$\frac{d(yI)}{dx} = Iq(x)$$

$$y = \frac{\int I(x)q(x)dx + c}{I(x)}$$

Bernoulli Equations

A Bernoulli differential equation has the form

$$y' + p(x)y = q(x)y^n \tag{2.24}$$

where n is a real number. The substitution

$$z = y^{1-n} \tag{2.25}$$

transforms 2.24 into a linear differential equation in the unknown function $z(x)$.

$$\frac{dy}{dx} = \frac{x^2 + 2}{y}$$

$$(x^2 + 2)dx - ydy = 0$$

$$\int (x^2 + 2)dx - \int ydy = c$$

$$\frac{1}{3}x^3 + 2x - \frac{1}{2}y^2 = c$$

$$y^2 = \frac{2}{3}x^3 + 4x + k$$

$$k = -2c$$

$$y = \sqrt{\frac{2}{3}x^3 + 4x + k}$$

$$y' = \frac{y + x}{x}$$

The homogeneous differential equation

$$\frac{dy}{dx} = f(x, y)$$

$$y = xv$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{xv + x}{x}$$

$$x\frac{dv}{dx} = 1$$

$$\int \frac{1}{x} dx - \int dv = c$$

$$v = \ln|x| - c$$

$$c = -\ln|k|$$

$$v = \ln|kx|$$

$$y = x\ln|kx|$$

$$2xydx + (1 + x^2)dy = 0$$

$$M(x, y) = 2xy$$

$$N(x, y) = 1 + x^2$$

$$\partial M / \partial y = \partial N / \partial x = 2x$$

$$\frac{\partial g(x, y)}{\partial x} = M(x, y)$$

$$\frac{\partial g(x, y)}{\partial y} = N(x, y)$$

$$\partial g / \partial x = 2xy$$

$$\int \frac{\partial g}{\partial x} dx = \int 2xydx \quad g(x, y) = x^2 y + h(y)$$

Note that when integrating with respect to x, the constant (*with respect to* x) of integration can depend on y.

We now determine $h(y)$

$$g(x, y) = x^2 y + h(y)$$

$$\partial g / \partial y = x^2 + h'(y)$$

$$\frac{\partial g(x, y)}{\partial y} = N(x, y)$$

$$x^2 + h'(y) = 1 + x^2$$

$$h'(y) = 1$$

$$h(y) = y + c_1$$

$$g(x, y) = x^2 y + y + c_1$$

$$y' + (4/x)y = x^4$$

$$y' + p(x)y = q(x)$$

$$I(x) = e^{\int p(x)dx}$$

$$\frac{d(yI)}{dx} = Iq(x)$$

$$y = \frac{\int I(x)q(x)dx + c}{I(x)}$$

$$\int p(x)dx = \int \frac{4}{x}dx = 4\ln|x| = \ln x^4$$

$$I(x) = e^{\int p(x)dx} = e^{\ln x^4} = x^4$$

$$x^4 y' + 4x^3 y = x^8 \quad \text{or} \quad \frac{d}{dx}(yx^4) = x^8$$

$$yx^4 = \frac{1}{9}x^9 + c \quad \text{or} \quad y = \frac{c}{x^4} + \frac{1}{9}x^5$$

$$y' + xy = xy^2$$

$z = y^{1-n}$ → into a linear differential equation

$$z = y^{1-2} = y^{-1}$$ → $$y = \frac{1}{z} \quad \text{and} \quad y' = -\frac{z'}{z^2}$$

$$y' + xy = xy^2$$ → $$-\frac{z'}{z^2} + \frac{x}{z} = \frac{x}{z^2} \quad \text{or} \quad z' - xz = -x$$ → $$y' + p(x)y = q(x)$$

$$I(x) = e^{\int(-x)dx} = e^{-x^2/2}$$

$$e^{-x^2/2}\frac{dz}{dx} - xe^{-x^2/2}z = -xe^{-x^2/2}$$

$$\frac{d}{dx}\left(ze^{-x^2/2}\right) = -xe^{-x^2/2}$$

$$\frac{d(yI)}{dx} = Iq(x)$$

$$ze^{-x^2/2} = e^{-x^2/2} + c$$

$$ze^{-x^2/2} = e^{-x^2/2} + c$$

$$z(x) = ce^{x^2/2} + 1$$

$$y = \frac{1}{z} = \frac{1}{ce^{x^2/2} + 1}$$

LINEAR DIFFERENTIAL EQUATIONS OF SECOND AND HIGHER ORDER WITH CONSTANT COEFFICIENTS

A differential equation of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + a_2 \frac{d^{n-2} y}{dx^{n-2}} + \ldots + a_n y = X \qquad \ldots(1)$$

where X is a function of x and $a_1, a_2, \ldots, a_n$ are constants is called a linear differential equation of n^{th} order with constant coefficients. Since the highest order of the derivative appearing in (1) is n, it is called a differential equation of n^{th} order and it is called linear.

Using the familiar notation of differential operators:

$$D = \frac{d}{dx}, \; D^2 = \frac{d^2}{dx^2}, \; D^3 = \frac{d^3}{dx^3} \ldots, D^n = \frac{d^n}{dx^n}$$

Then (1) can be written in the form

Then (1) can be written in the form

$$\{D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots a_n\} y = X$$

i.e., $$f(D)\, y = X$$

where $$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots a_n.$$

Here $f(D)$ is a polynomial of degree n in D

SOLUTION OF A HOMOGENEOUS SECOND ORDER LINEAR DIFFERENTIAL EQUATION

We consider the homogeneous equation

$$\frac{d^2y}{dx^2} + p\frac{dy}{dx} + qy = 0$$

where p and q are constants

$$(D^2 + pD + q)\,y = 0$$

The Auxiliary equations (A.E.) put $D = m$

$$m^2 + pm + q = 0$$

There are three cases.

Case (*i*): Roots are real and distinct

The roots are real and distinct, say m_1 and m_2 *i.e.*, $m_1 \neq m_2$

Hence, the general solution of eqn. (1) is

$$y = C_1\,e^{m_1x} + C_2\,e^{m_2x}$$

Case (*ii*): Roots are equal

The roots are equal *i.e.*, $m_1 = m_2 = m$.

Hence, the general solution of eqn. (1) is

$$y = (C_1 + C_2\,x)\,e^{mx}$$

Case (*iii*): Roots are complex

The Roots are complex, say $\alpha \pm i\beta$

Hence, the general solution is

$$y = e^{\alpha x}\,(C_1 \cos \beta x + C_2 \sin \beta x)$$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0.$$

Given equation is $(D^2 - 5D + 6)\, y = 0$

$$(m-2)\,(m-3) = 0$$
$$m = 2,\ 3$$
$$m_1 = 2,\ m_2 = 3$$

The roots are real and distinct

The general solution of the equation is

$$y = C_1\, e^{2x} + C_2\, e^{3x}.$$

$$\frac{d^3y}{dx^3} - \frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$$

Given equation is $(D^3 - D^2 - 4D + 4)\, y = 0$

A.E. is $m^3 - m^2 - 4m + 4 = 0$

$$\begin{aligned} m^2(m-1) - 4(m-1) &= 0 \\ (m-1)(m^2-4) &= 0 \\ m &= 1, m = \pm 2 \\ m_1 &= 1, m_2 = 2, m_3 = -2 \end{aligned}$$

∴ The general solution of the given equation is

$$y = C_1 e^x + C_2 e^{2x} + C_3 e^{-2x}.$$

$$\frac{d^2y}{dx^2} + 8\frac{dy}{dx} + 16y = 0$$

$$(D^2 + 8D + 16)\,y = 0$$

A.E. is $m^2 + 8m + 16 = 0$

$$\therefore \quad (m + 4)^2 = 0$$

$$(m + 4)\,(m + 4) = 0$$

$$m = -4, -4$$

$\therefore$ The general solution is

$$y = (C_1 + C_2\,x)\,e^{-4x}.$$

$$\frac{d^2y}{dx^2} + w^2y = 0$$

$$(D^2 + w^2)\,y = 0$$

A.E. is $m^2 + w^2 = 0$

$$m^2 = -w^2 = w^2i^2 \quad (i^2 = -1)$$

$$m = \pm\, w\, i$$

This is the form $\alpha \pm i\beta$ where $\alpha = 0$, $\beta = w$.

$\therefore$ The general solution is

$$y = e^{0t}\,(C_1 \cos wt + C_2 \sin wt)$$

$\therefore$ $y = C_1 \cos wt + C_2 \sin wt$.

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 13y = 0$$

$$(D^2 + 4D + 13)\, y = 0$$

A.E. is $m^2 + 4m + 13 = 0$

$$m = \frac{-4 \pm \sqrt{16 - 52}}{2}$$

$$= -2 \pm 3i \text{ (of the form } \alpha \pm i\beta)$$

∴ The general solution is

$$y = e^{-2x}\,(C_1 \cos 3x + C_2 \sin 3x).$$

TEMPERATURE PROBLEMS

Newton's law of cooling, which is equally applicable to heating, states that *the time rate of change of the temperature of a body is proportional to the temperature difference between the body and its surrounding medium.*

$$dT/dt = -k(T - T_m)$$

GROWTH AND DECAY PROBLEMS

Let $N(t)$ denote the amount of substance (or population) that is either growing or decaying. If we assume that dN/dt, the time rate of change of this amount of substance, is proportional to the amount of substance present, then $dN/dt = kN$, or

$$\frac{dN}{dt} - KN = 0$$

FALLING BODY PROBLEMS

Consider a vertically falling body of mass m that is being influenced only by gravity g and an air resistance that is proportional to the velocity of the body. Assume that both gravity and mass remain constant and, for convenience, choose the downward direction as the positive direction.

Newton's second law of motion: *The net force acting on a body is equal to the time rate of change of the momentum of the body; or, for constant mass,*

$$F = m\frac{dv}{dt}$$

where F is the net force on the body and v is the velocity of the body, both at time t.

A bacteria culture is known to grow at a rate proportional to the amount present. After one hour, 1000 strands of the bacteria are observed in the culture; and after four hours, 3000 strands. Find (*a*) an expression for the approximate number of strands of the bacteria present in the culture at any time t and (*b*) the approximate number of strands of the bacteria originally in the culture.

$$dN/dt - kN = 0$$

$$N(t) = ce^{kt}$$

At $t = 1$, $N = 1000$; hence,

$$1000 = ce^{k}$$

At $t = 4$, $N = 3000$; hence,

$$3000 = ce^{4k}$$

$$k = \frac{1}{3}\ln 3 = 0.366 \quad \text{and} \quad c = 1000e^{-0.366} = 694$$

The population of a certain country is known to increase at a rate proportional to the number of people presently living in the country. If after two years the population has doubled, and after three years the population is 20,000, estimate the number of people initially living in the country.

$$\frac{dN}{dt} - kN = 0$$

$$N = ce^{kt}$$

$$N = N_0\, e^{kt}$$

$$2N_0 = N_0 e^{2k} \quad \text{from which} \quad k = \frac{1}{2}\ln 2 = 0.347$$

$$N = N_0 e^{0.347t}$$

At $t = 3$, $N = 20{,}000$. Substituting these values into (3), we obtain

$$20{,}000 = N_0 e^{(0.347)(3)} = N_0(2.832) \qquad \text{or} \qquad N_0 = 7062$$

Five mice in a stable population of 500 are intentionally infected with a contagious disease to test a theory of epidemic spread that postulates the rate of change in the infected population is proportional to the product of the number of mice who have the disease with the number that are disease free. Assuming the theory is correct, how long will it take half the population to contract the disease?

$$\frac{dN}{dt} = kN(500 - N)$$

$$\frac{dN}{N(500-N)} - k\,dt = 0$$

Fracciones Parciales

$$\frac{1}{N(500-N)} = \frac{1/500}{N} + \frac{1/500}{500-N}$$

$$\frac{1}{500}\left(\frac{1}{N} + \frac{1}{500-N}\right)dN - k\,dt = 0$$

Its solution is

$$\frac{1}{500}\int\left(\frac{1}{N} + \frac{1}{500-N}\right)dN - \int k\,dt = c$$

or

$$\frac{1}{500}(\ln|N| - \ln|500-N|) - kt = c$$

which may be rewritten as

$$\ln\left|\frac{N}{500-N}\right| = 500(c + kt)$$

$$\frac{N}{500-N} = e^{500(c+kt)}$$

$$\frac{N}{500-N} = c_1 e^{500kt}$$

$$\frac{5}{495} = c_1 e^{500k(0)} = c_1$$

$$\frac{N}{500-N} = \frac{1}{99}e^{500kt}$$

We seek a value of t when $N = 250$.

$$1 = \frac{1}{99}e^{500kt}$$

A metal bar at a temperature of 100° F is placed in a room at a constant temperature of 0° F. If after 20 minutes the temperature of the bar is 50° F, find (*a*) the time it will take the bar to reach a temperature of 25° F and (*b*) the temperature of the bar after 10 minutes.

$dT/dt = -k(T - T_m)$ $\qquad$ $T_m = 0$

$$\frac{dT}{dt} + kT = 0$$

$$T = ce^{-kt}$$

$$T = 100e^{-kt}$$

$$50 = 100e^{-20k} \quad \text{from which} \quad k = \frac{-1}{20}\ln\frac{50}{100} = \frac{-1}{20}(-0.693) = 0.035$$

$$T = 100e^{-0.035t}$$

a) $t = 39.6 \text{ min}$

b) $T = 100e^{(-0.035)(10)} = 100(0.705) = 70.5° \text{ F}$

A body of mass m is thrown vertically into the air with an initial velocity v_0. If the body encounters an air resistance proportional to its velocity, find (a) the equation of motion in the coordinate system of Fig. 7-6, (b) an expression for the velocity of the body at any time t, and (c) the time at which the body reaches its maximum height.

$$F = m\frac{dv}{dt}$$

$$\frac{dv}{dt} + \frac{k}{m}v = -g$$

$$v = ce^{-(k/m)t} - mg/k$$

At $t = 0,\ v = v_0$

$$c = v_0 + (mg/k)$$

$$v = \left(v_0 + \frac{mg}{k}\right)e^{-(k/m)t} - \frac{mg}{k}$$

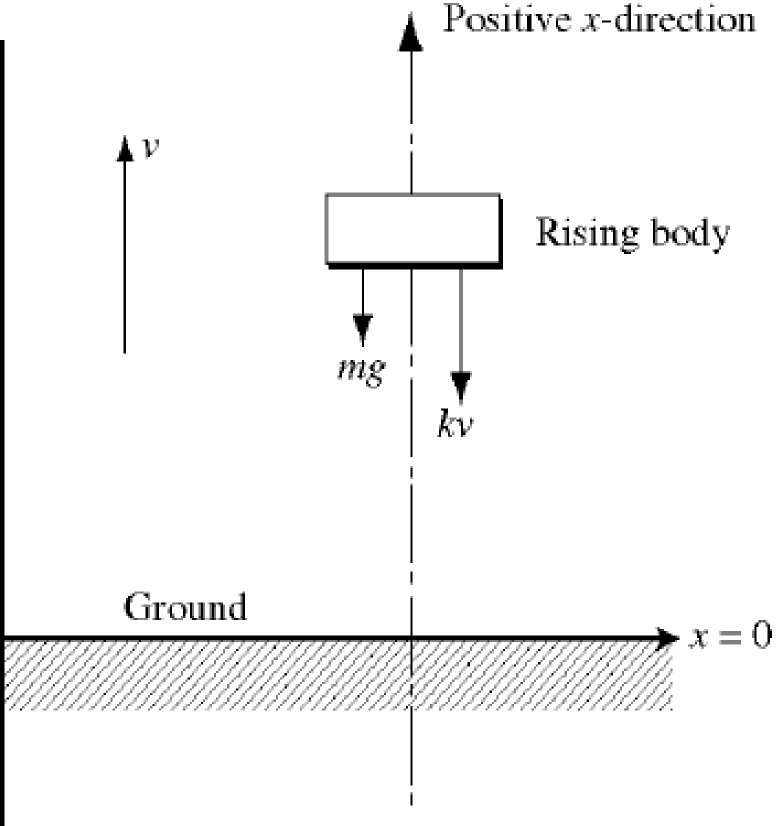


Fig. 7-6

$$0 = \left(v_0 + \frac{mg}{k}\right)e^{-(k/m)t} - \frac{mg}{k}$$

$$e^{-(k/m)t} = \frac{1}{1 + \frac{v_0 k}{mg}}$$

$$-(k/m)t = \ln\left(\frac{1}{1 + \frac{v_0 k}{mg}}\right)$$

$$t = \frac{m}{k}\ln\left(1 + \frac{v_0 k}{mg}\right)$$

A 50-gal tank initially contains 10 gal of fresh water. At $t = 0$, a brine solution containing 1 lb of salt per gallon is poured into the tank at the rate of 4 gal/min, while the well-stirred mixture leaves the tank at the rate of 2 gal/min. Find (*a*) the amount of time required for overflow to occur and (*b*) the amount of salt in the tank at the moment of overflow.

$$\frac{dQ}{dt} + \frac{2}{10+2t}Q = 4$$

$$I(t) = e^{\int [2/(10+2t)]\,dt} = e^{\ln[10+2t]} = 10 + 2t$$

$$(10+2t)\frac{dQ}{dt} + 2Q = 40 + 8t$$

$$\frac{d}{dt}[(10+2t)Q] = 40 + 8t$$

$$(10+2t)Q = 40t + 4t^2 + c$$

$$10 + 2t = 50$$

$$t = 20 \text{ min}$$

$$Q(t) = \frac{40t + 4t^2 + c}{10+2t}$$

$$Q = \frac{40(20) + 4(20)^2}{10 + 2(20)} = 48 \text{ lb}$$

Reversible Isothermal Process in a Perfect Gas

Calculation of q, w, and ΔU

Consider the special case of a reversible isothermal (constant-T) process in a perfect gas. (Throughout this section, the system is assumed closed.) For a fixed amount of a

$$w = -\int_1^2 P\,dV = -\int_1^2 \frac{nRT}{V}\,dV = -nRT\int_1^2 \frac{1}{V}\,dV = -nRT(\ln V_2 - \ln V_1)$$

$$w = -q = nRT \ln \frac{V_1}{V_2} = nRT \ln \frac{P_2}{P_1} \qquad \text{rev. isothermal proc., perf. gas}$$